Math 564: Real analysis and measure theory
Lecture 3

Observation. Let $(X, \mathcal{S})$ be a measurable space.

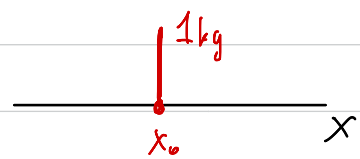
(a) Any cfbl nonnegative linear combination of measures on $(X, \mathcal{S})$ is a measure, i.e. if μ_n are measures on $(X, \mathcal{S})$ and $a_n \geq 0$, then $\sum_{n \in \mathbb{N}} a_n \mu_n$ is a measure.

(b) Any convex combin. of probability meas. on $(X, \mathcal{S})$ is a prob. meas., i.e. if μ_n are prob. measures and $a_n \geq 0$ with $\sum_{n \in \mathbb{N}} a_n = 1$, then $\sum_{n \in \mathbb{N}} a_n \mu_n$ is a prob. meas.

Examples. (a) The zero measure: on any set X, the zero measure is $z : \mathcal{P}(X) \to \{0\}$.

(b) The Dirac (delta) measure / point measure: Let X be a set, fix a point $x_0 \in X$.
Define a measure $\delta_{x_0} : \mathcal{P}(X) \to [0,1]$ by

$$\delta_{x_0}(B) := \begin{cases} 1 & \text{if } x_0 \in B \\ 0 & \text{o.w.} \end{cases}$$



This is called the Dirac measure at x_0.

(c) Counting measure: on any set X, the counting measure $\mathcal{X} : \mathcal{P}(X) \to [0, \infty]$ is given by

$$\mathcal{X}(B) := \begin{cases} |B| & \text{if } B \text{ is finite.} \\ \infty & \text{o.w.} \end{cases}$$

Note that when X is cfbl, then $\mathcal{X} = \sum_{x \in X} \delta_x$.

Also, $\mathcal{X}$ is finite when X is finite

σ-finite   cfbl

not σ-finite   uncfbl

(d) Given a set X, define a measure μ on the σ-algebra of cfbl and co-cfbl subsets of X as follows:

$$\mu(B) := \begin{cases} 0 & \text{if } B \text{ is cfbl} \\ 1 & \text{o.w.} \end{cases}$$

If X is cftbl then μ is the zero measure.

Def. Let (X, Σ, μ) be a measure space. A set $B \in \Sigma$ is called an atom (or μ-atom) if $\mu(B) > 0$ and for all $A \subseteq B$, $A \in \Sigma$, either $\mu(A) = 0$ or $\mu(A) = \mu(B)$.
A measure space (X, Σ, μ) is called
- atomic (or purely atomic) if every positive measure set in Σ contains an atom.
- atomless if there are no atoms.

Caution. The zero measure is both atomic and atomless.

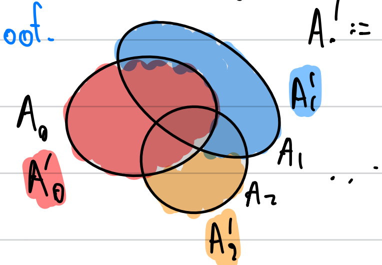
All measures in examples (a)–(d) are atomic. To define interesting atomless meas., we need to first define them on an algebra $\mathcal{A}$ and then extend them to the σ-algebra generated by $\mathcal{A}$.

Finitely additive measures and premeasures.

Def. Let $\mathcal{A}$ be an algebra on a set X. A function $\mu : \mathcal{A} \to [0, \infty]$ is called a finitely additive measure (resp. a cftbly additive measure or premeasure) if $\mu(\emptyset) = 0$ and μ is finitely (resp. cftbly) additive.

Disjointification trick. For an algebra $\mathcal{A}$, any cftbl union $\bigcup_{n \in \mathbb{N}} A_n$ of sets $A_n \in \mathcal{A}$ is equal to a cftbl disjoint union $\bigsqcup_{n \in \mathbb{N}} A'_n$ of sets $A'_n \in \mathcal{A}$.

Proof. $A'_0 := A_0$, $A'_1 := A_1 \setminus A_0$, $A'_2 := A_2 \setminus (A_0 \cup A_1)$, ..., $A'_n := A_n \setminus \bigcup_{i<n} A_i$. $\square$



Properties of finitely additive measures. Let μ be a finitely additive measure on an algebra $\mathcal{A}$ on on a set X. Then:

(a) μ is monotone: if $A \subseteq B$ then $\mu(A) \leq \mu(B)$ for all $A, B \in \mathcal{A}$.

(b) μ is ctbly superadditive: $\mu\left(\bigsqcup_{n \in \mathbb{N}} A_n\right) \geq \sum_{n \in \mathbb{N}} \mu(A_n)$ for all $A_n \in \mathcal{A}$ with $\bigsqcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$.

(c) μ is finitely subadditive*: $\mu\left(\bigcup_{n < N} A_n\right) \leq \sum_{n < N} \mu(A_n)$ for all $A_n \in \mathcal{A}$.

If μ is moreover ctbly additive (i.e. a premeasure), then it is ctbly subadditive*:

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu(A_n) \quad \text{for all } A_n \in \mathcal{A} \text{ with } \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}.$$

Proof. (a) $\mu(B) = \mu((B \setminus A) \sqcup A) \overset{\text{fin. add.}}{=} \mu(B \setminus A) + \mu(A) \geq \mu(A)$.

(b) $\mu\left(\bigsqcup_{n \in \mathbb{N}} A_n\right) = \mu\left(A_0 \sqcup A_1 \sqcup \ldots \sqcup A_N \sqcup \bigsqcup_{n > N} A_n\right) \overset{\text{fin. add.}}{=} \sum_{n \leq N} \mu(A_n) + \mu\left(\bigsqcup_{n > N} A_n\right) \geq \sum_{n \leq N} \mu(A_n) \xrightarrow[N \to \infty]{}$

$\to \sum_{n \in \mathbb{N}} \mu(A_n)$.

(c) By the disjointification trick: $\mu\left(\bigcup_{n < N} A_n\right) = \mu\left(\bigsqcup_{n < N} A_n'\right) \overset{\text{fin. add.}}{=} \sum_{n < N} \mu(A_n') \overset{\text{monotonicity}}{\leq} \sum_{n < N} \mu(A_n)$.
Same for ctbl unions if μ is ctbly additive $\square$

Constructions of premeasures.

Bernoulli premeasures. Let $X = 2^{\mathbb{N}}$. Any prob. measure on $2 := \{0, 1\}$ is of the form $\nu_p(1) = p$ and $\nu_p(0) = 1 - p$ for some $p \in (0,1)$. Fixing $p \in (0,1)$, we will define a premeasure μ_p on the algebra $\mathcal{A}$ of clopen sets of $2^{\mathbb{N}}$ satisfying $\mu_p([w]) = \nu_p(1)^{(\# \text{ of } 1\text{s in } w)} \nu_p(0)^{(\# \text{ of } 0\text{s in } w)}$.
This premeasure μ_p is also denoted $\nu_p^{\mathbb{N}}$ and is called the Bernoulli(p) premeasure.

Recall (from HW) that sets in $\mathcal{A}$ are exactly finite disjoint unions of cylinders, so we first define $\tilde{\mu}_p$ on a cylinder $[w]$, $w \in 2^{<\mathbb{N}}$, by

$$\tilde{\mu}_p([w]) = p^{(\#\text{ of } 1\text{s in } w)} \cdot (1-p)^{(\#\text{ of } 0\text{s in } w)} = \nu_p(1)^{(\#\text{ of } 1\text{s in } w)} \cdot \nu_p(0)^{(\#\text{ of } 0\text{s in } w)},$$

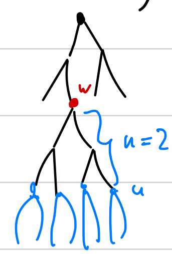
e.g. $\tilde{\mu}_p([01100]) = p^2 \cdot (1-p)^3$. Then for each $B \in \mathcal{A}$, we "define"

$$\mu_p(B) :\simeq \sum_{n<N} [w_n],$$

where $B = \bigsqcup_{n<N} [w_n]$. We need to show that this is well-defined, i.e. doesn't depend on how B is written as a disjoint union of cylinders.

Claim (a). $\tilde{\mu}_p$ is finitely additive on equal-length cylinders, i.e. for any cylinder $[w]$ and $n \in \mathbb{N}$,

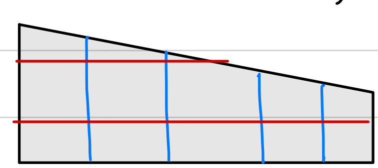
$$\tilde{\mu}_p([w]) = \sum_{u \in 2^n} \tilde{\mu}_p([wu]).$$



Proof. By induction on n, it enough to verify for $n=1$: $\tilde{\mu}_p([w0]) + \tilde{\mu}_p([w1]) = \tilde{\mu}_p([w]) \cdot (1-p) + \tilde{\mu}_p([w]) \cdot p = \tilde{\mu}_p([w])$. $\square$

Claim (b). Let $A \in \mathcal{A}$ and $\mathcal{P}_1, \mathcal{P}_2$ be two finite partitions of A into cylinders. Then

$$\sum_{P_1 \in \mathcal{P}_1} \tilde{\mu}_p(P_1) = \sum_{P_2 \in \mathcal{P}_2} \tilde{\mu}_p(P_2).$$



Proof. Let $\mathcal{Q}$ be a common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$ (still finite), and we take $\mathcal{Q}$ so that all cylinders in $\mathcal{Q}$ have the same length (by splitting each cylinder into a finite partition of cylinders of bigger length). Then

$$\sum_{P_1 \in \mathcal{P}_1} \tilde{\mu}_p(P_1) \underset{\text{Claim (a)}}{=} \sum_{P_1 \in \mathcal{P}_1} \sum_{\substack{Q \in \mathcal{Q} \\ Q \subseteq P_1}} \tilde{\mu}_p(Q) = \sum_{Q \in \mathcal{Q}} \tilde{\mu}_p(Q) = \sum_{P_2 \in \mathcal{P}_2} \sum_{\substack{Q \in \mathcal{Q} \\ Q \subseteq P_2}} \tilde{\mu}_p(Q) \underset{\text{Claim (a)}}{=} \sum_{P_2 \in \mathcal{P}_2} \tilde{\mu}_p(P_2).$$

$\square$

Claim (b) show the well-definedness of μ_p on $\mathcal{A}$. It also implies that:

Cor. μ_p is finitely additive.

Proof. Immediate from Claim (b), HW.

Claim (c). μ_p is ctbly additive (i.e. a premeasure).

Proof. This is automatic by compactness: if a clopen set C (hence closed, hence compact) is a disjoint union of clopen (hence open) sets $\bigsqcup_{n\in\mathbb{N}} U_n$, then all but finitely many of these clopen sets have to be empty (because there is a finite subcover of $\{U_n\}_{n\in\mathbb{N}}$). $\square$

This construction equally works for $A^{\mathbb{N}}$, for any finite nonempty A, and any prob. measure ν on A (instead of ν_p on $2 := \{0,1\}$), and we obtain a premeasure $\mu := \nu^{\mathbb{N}}$ on the algebra $\mathcal{A}$ of clopen subsets of $A^{\mathbb{N}}$ satisfying

$$\mu([w]) = \nu(w_0) \cdot \nu(w_1) \cdot \ldots \cdot \nu(w_{n-1}),$$

for every $w \in A^n$ and $n \in \mathbb{N}$.